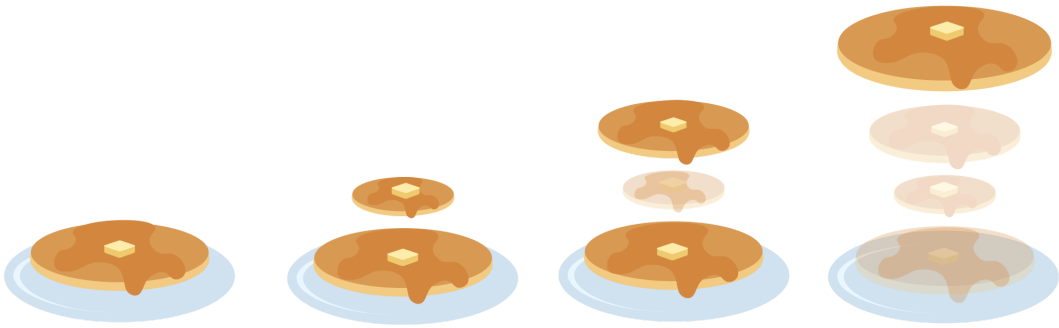
Code Jam 2021 - Round 2

Hidden Pancakes

Problem

We are cooking $\mathbf{N}$ pancakes in total. We cook one pancake with a 1 centimeter (cm) radius, one with a 2 cm radius, one with a 3 cm radius, ..., and one with an $\mathbf{N}$ cm radius, not necessarily in that order. After we cook the first pancake, we just lay it on a plate. After we cook each subsequent pancake, we lay it on top of the previously made pancake, with their centers coinciding. In this way, a pancake is visible from the top of the stack when we first add it. A pancake only becomes hidden if we later cook another pancake with a larger radius.

For example, say we cook 4 pancakes. We first cook the pancake with radius 3 cm, and it is visible. Then, we cook the pancake with radius 1 cm, lay it on top of the first one and both are visible. Third, we cook the pancake with radius 2 cm, and now that covers the previous pancake, but not the first one, so 2 pancakes remain visible in total. Finally, we cook the pancake with radius 4 cm which covers the other pancakes leaving only 1 visible pancake. The picture below illustrates the state of the stack after each pancake is cooked. Within each stack, the fully colored pancakes are visible and the semi-transparent pancakes are not visible.



Let $\mathbf{V_i}$ be the number of visible pancakes when the stack contains exactly i pancakes. In the example above, $\mathbf{V_1} = 1$, $\mathbf{V_2} = 2$, $\mathbf{V_3} = 2$, and $\mathbf{V_4} = 1$.

Given the list $\mathbf{V_1}, \mathbf{V_2}, \ldots, \mathbf{V_N}$, how many of the $\mathbf{N}!$ possible cooking orders yield those values? Since the output can be a really big number, we only ask you to output the remainder of dividing the result by the prime $10^9 + 7$ (1000000007).

Input

The first line of the input gives the number of test cases, $\mathbf{T}$. $\mathbf{T}$ test cases follow, each described with two lines. The first line of a test case contains a single integer $\mathbf{N}$, the number of pancakes we cook. The second line of a test case contains $\mathbf{N}$ integers $\mathbf{V_1}$, $\mathbf{V_2}$, ..., $\mathbf{V_N}$, representing the number of visible pancakes after we cook 1, 2, ..., $\mathbf{N}$ pancakes, respectively.

Output

For each test case, output one line containing `Case #`x`:` y, where x is the test case number (starting from 1) and y is the number of cooking orders of $\mathbf{N}$ pancakes that yield the given numbers of visible pancakes after each step, modulo the prime $10^9 + 7$ (1000000007).

Limits

Memory limit: 1 GB.
$1 \leq \mathbf{T} \leq 100$.
$1 \leq \mathbf{V_i} \leq i$, for all i.

Test Set 1 (Visible Verdict)

Time limit: 30 seconds.
$2 \leq \mathbf{N} \leq 13$.

Test Set 2 (Hidden Verdict)

Time limit: 40 seconds.
$2 \leq \mathbf{N} \leq 10^5$.

Sample

Note: there are additional samples that are not run on submissions down below.

Sample Input

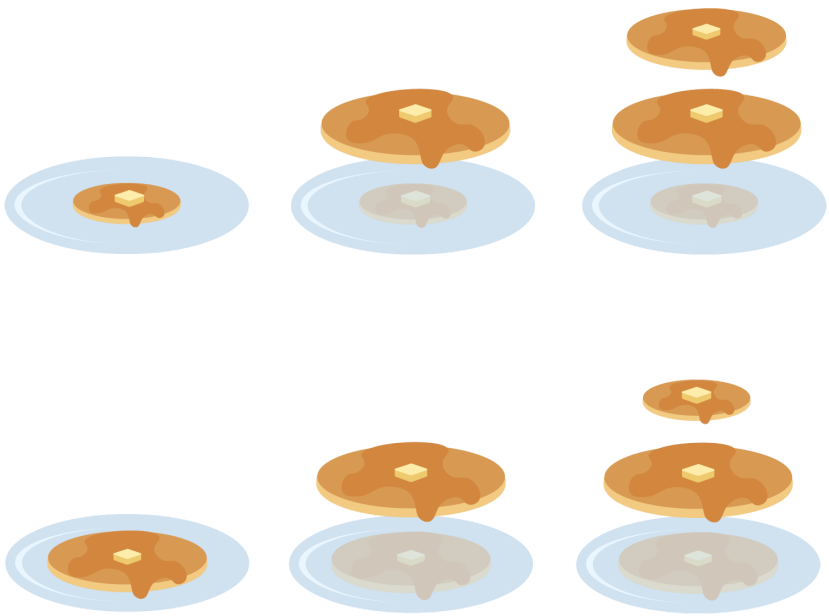
```
3
4
1 2 2 1
3
1 1 2
3
1 1 3
```

Sample Output

```
Case #1: 1
Case #2: 2
Case #3: 0
```

Sample Case #1 is explained in the problem statement. The order $3, 1, 2, 4$ is the only one that yields the given $\mathbf{V_i}$s.

In Sample Case #2, both the order $1, 3, 2$ and the order $2, 3, 1$ yield the intended $\mathbf{V_i}$s. The pictures below illustrate both options.



In Sample Case #3, only 1 pancake is visible after the second is made, so there is no way to have more than 2 visible pancakes by only adding a third.

Additional Sample - Test Set 2

The following additional sample fits the limits of Test Set 2. It will not be run against your submitted solutions.

Sample Input

```
1
24
1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2
1 2 1 2 1 2 1 2
```

Sample Output

```
Case #1: 234141013
```

In the Sample Case for Test Set 2, there are 316234143225 cooking orders that yield the given $\mathbf{V_i}$s. Modulo $10^9 + 7$, this value is 234141013.